Bäcklund transformations for difference Hirota equation and supersymmetric Bethe ansatz*

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Abstract

We consider $GL(K|M)$-invariant integrable supersymmetric spin chains with twisted boundary conditions and elucidate the role of Bäcklund transformations in solving the difference Hirota equation for eigenvalues of their transfer matrices. The nested Bethe ansatz technique is shown to be equivalent to a chain of successive Bäcklund transformations "undressing" the original problem to a trivial one.

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1 Introduction

For integrable models, the relationship between classical and quantum systems is in no way exhausted by their correspondence in the classical limit. As is now well known, classical integrable equations often appear in quantum integrable problems as exact relations even for $\hbar \neq 0$.

One important example of this general phenomenon was investigated in [1, 2, 3], where it was shown that the spectrum of commuting transfer matrices (integrals of motion) in quantum integrable models can be found in terms of discrete classical dynamics, also integrable, defined in the space whose points label the commuting transfer matrices. For integrable $GL(K)$-invariant spin chains coordinates in this space are parameters specifying finite-dimensional irreducible representations of the group $GL(K)$ and the spectral parameter. The classical dynamics in this space is generated by functional relations for the transfer matrices established in [4, 5, 6] for the ordinary bosonic case and extended to the supersymmetric case in [7]. Among them the most important is the bilinear functional equation for the eigenvalues of the transfer matrices (T-functions) which has the form of the Hirota bilinear difference equation [8]. For brevity, we call it the TT-relation. It is the starting point of our approach.

The Hirota equation is probably the most famous equation in the theory of classical integrable systems on the lattice. It provides a universal integrable discretization of various soliton equations and, at the same time, is a generating equation for their hierarchies. It is involved in a large body of integrable problems, classical and quantum. Like other soliton equations, the Hirota equation admits (auto) Bäcklund transformations, i.e., transformations that send any solution to another solution of the same equation. They allow one to construct a family of solutions that are connected with a particularly simple one by a finite chain of such transformations.

The Bäcklund transformations play a central role in our method serving as an alternative to the standard Bethe ansatz technique. The nested Bethe ansatz solution of $GL(K)$-invariant spin chains consists in successive increasing the rank of the group by applying the Bethe ansatz repeatedly. In this way, one can descend from $GL(K)$ to $GL(K-1)$ until the problem gets trivialized at $K = 0$. At intermediate stages of this procedure, one introduces a number of auxiliary "Q-functions" (eigenvalues of Baxter's Q-operators) connected with the T-functions by Baxter's TQ-relations. Their zeros with respect to the spectral parameter obey the system of Bethe equations. This purely quantum technique has a remarkable classical interpretation [1, 2]: it is equivalent to a chain of Bäcklund transformations for the Hirota equation. The TQ-relations appear then as a constituent of auxiliary linear problems for the Hirota equation. The rank of the group becomes an additional discrete variable, with the dependence on this variable being again described by the Hirota-like equation. Since the solutions are polynomials in the spectral parameter, their zeros obey equations of motion of a finite-dimensional dynamical system in discrete time. The equations of motion are just Bethe equations.

Recently, this approach was applied [9] to supersymmetric spin chains constructed by means of $GL(K|M)$-invariant solutions to the graded Yang-Baxter equation [10, 11]. In this case, there are two rather then one additional discrete flows corresponding to the bosonic and fermionic ranks, K and M. Their consistency leads to a non-trivial

bilinear relation for eigenvalues of Baxter's Q-operators (the QQ-relation). In the present paper we extend these results to $GL(K|M)$-invariant spin chains with twisted (quasi-periodic) boundary conditions. The twisting parameters enter the solution as continuous parameters of the Bäcklund transformations.

It is implied that the reader is familiar with the standard notions and facts related to supergroups and their representations [12, 13, 14], as well as with the quantum inverse scattering method constructions [15, 16, 17].

2 The TT-relation

Let us recall the construction of the family of commuting transfer matrices in integrable spin chains. It is basically the same for ordinary and supersymmetric models. We consider lattice models (spin chains) with the symmetry supergroup $GL(K|M)$ constructed by means of $GL(K|M)$-invariant R-matrices. Such R-matrices depend on the spectral parameter $u \in \mathbb{C}$ and act in the tensor product $V_0 \otimes V_1$ of two linear spaces, where irreducible representations π_0 and π_1 of $GL(K|M)$ are defined. The space V_0 is usually called auxiliary space and V_1 (local) quantum space. For our purposes we need the case when π_1 is the vector representation, i.e., $V_1 = V = \mathbb{C}^K \oplus \mathbb{C}^M$ while π_0 is an arbitrary tensor representation of the supergroup. The R-matrix reads

$$R_{01}(u) = u + 2\sum_{\alpha\beta}(-1)^{p(\beta)}\pi_0(E_{\alpha\beta}) \otimes e_{\beta\alpha}\,. \tag{1}$$

Here $p(\beta)$ is parity of the index β ($p(\beta) = 0$ or 1), $e_{\alpha\beta}$ are matrices with the entries $(e_{\alpha\beta})_{\alpha'\beta'} = \delta_{\alpha\alpha'}\delta_{\beta\beta'}$, and $\pi_0(E_{\alpha\beta})$ are generators of the $gl(K|M)$ superalgebra in the representation π_0. The first (scalar) term is to be understood as u multiplied by the unity matrix $\pi_0(I) \otimes I_{V_1}$, where $I \in GL(K|M)$ is the unity element in the group and I_{V_1} is the identity operator in the space V_1. This R-matrix is the $GL(K|M)$-invariant solution to the graded Yang-Baxter equation [10, 11]. The supergroup invariance means that

$$\pi_0(g) \otimes \pi_1(g)\, R_{01}(u) = R_{01}(u)\, \pi_0(g) \otimes \pi_1(g) \tag{2}$$

for any $g \in GL(K|M)$.

In order to introduce generalized integrable spin chains on N sites, take N copies of the space $V = V_1 = V_2 = \ldots = V_N$ (one for each site of the chain) and the corresponding R-matrices $R_{0i}(u)$ acting in $V_0 \otimes V_i$. The Hilbert space of states of the model, $\mathcal{H}$, is the tensor product of local quantum spaces V_i over all sites of the chain: $\mathcal{H} = \otimes_{i=1}^N V_i$. We will call $\mathcal{H}$ the quantum space of the model. The quantum monodromy matrix is constructed as the product of the R-matrices $R_{0i}(u)$ in the space V_0:

$$\mathcal{T}(u) = R_{01}(u-\xi_1)R_{02}(u-\xi_2)\ldots R_{0N}(u-\xi_N)\,. \tag{3}$$

The quantities ξ_i are input data characterizing the (inhomogeneous) spin chain. The supertrace of the quantum monodromy matrix taken in the space V_0 gives a family of operators in the quantum space depending on u and π_0 (called transfer matrices), which mutually commute for any values of these parameters. A more general construction

involves twisted (quasi-periodic) boundary conditions defined by means of a diagonal matrix

$$g = \mathrm{diag}\,(x_1, \dots, x_K,\, y_1, \dots, y_M) \in GL(K|M) \tag{4}$$

(for simplicity, we identify elements of the supergroup $GL(K|M)$ with matrices from its vector representation). Then the commuting family of transfer matrices is given by the formula

$$T^{(\pi_0)}(u; g) = \mathrm{str}_{\pi_0}\,(\pi_0(g)\mathcal{T}(u))\;. \tag{5}$$

The graded Yang-Baxter equation combined with the $GL(K|M)$ invariance implies that they commute for different u and π_0 (but not for different g!): $[T^{(\pi_0)}(u; g),\, T^{(\pi'_0)}(u'; g)] = 0$.

Below in this paper we will be especially interested in the case when the representations π_0 are "rectangular", i.e. correspond to rectangular Young diagram. Given a rectangular diagram of length s and height a, let π_s^a be the corresponding representation. We define the quantum transfer matrices $T(a, s, u)$ for rectangular representations in the auxiliary space by the formula

$$T(a, s, u) = \mathrm{str}_{\pi_s^a}\Big(\pi_s^a(g)\mathcal{T}(u-s+a)\Big)\,. \tag{6}$$

It differs from (5) by the shift of u which is convenient in what follows. As a rule, we will not indicate the dependence on g explicitly. One may formally extend this definition to zero values of a and s which correspond to the trivial representation $\pi_0^a = \pi_s^0 = \pi_\emptyset$ ($\pi_\emptyset(g) = 1$ for any $g \in GL(K|M)$). Taking into account that $\pi_\emptyset(E_{\alpha\beta}) = 0$ for all generators of the superalgebra, we conclude from formulas (1), (3) and (6) that

$$T(0, s, u) = \prod_{j=1}^{N}(u - s - \xi_j)\,, \quad T(a, 0, u) = \prod_{j=1}^{N}(u + a - \xi_j) \tag{7}$$

where ξ_j are the same as in (3). So we see that $T(0, s, u)$ and $T(a, 0, u)$ are unity operators in the quantum space multiplied by scalar polynomial functions. These functions are fixed input data of the problem. We put

$$\phi(u) = \prod_{i=1}^{N}(u - \xi_i)\,, \tag{8}$$

then $T(0, s, u) = \phi(u - s)$, $T(a, 0, u) = \phi(u + a)$.

The transfer matrices constructed above are linearly independent but are connected by non-linear functional relations. In particular, the transfer matrices (6) for rectangular representations are known [5, 6] to obey the TT-relation

$$T(a, s, u+1)T(a, s, u-1) = T(a, s+1, u)T(a, s-1, u) + T(a+1, s, u)T(a-1, s, u). \tag{9}$$

This is the famous Hirota bilinear difference equation [8], where T plays the role of the τ-function. Since all the transfer matrices mutually commute, they can be simultaneously diagonalized by a u-independent similarity transformation, and thus the same relation is valid for any of their eigenvalues. Keeping this in mind, we will think of the transfer matrices as scalar functions and call them T-functions. Our strategy is to treat equation (9) as the basic equation of the quantum theory trying to derive the results for the

spectrum of quantum transfer matrices from it. To do this, we need to specify the boundary conditions and analytic properties of the solutions. The boundary conditions will be discussed in section 4. The analytic properties in u are determined by the type of the R-matrix (see a more detailed discussion in [3]). For quantum spin chains with polynomial R-matrices (1) all the T-functions $T(a,s,u)$ are polynomials in u. Thus we are led to the study of polynomial solutions to the Hirota equation.

One should note that the general solution to the Hirota equation with the boundary T-functions at $a=0$ and $s=0$ as in (7) is given by the Bazhanov-Reshetikhin determinant formula [4]. In our normalization it has the form

$$T(a,s,u) = H^{-1}(u-s,a) \det_{1\le i,j\le a} T(1,\, s+i-j,\, u+a+1-i-j)\,, \tag{10}$$

where the function $H(u,a)$ is defined as follows: $H(u,0) = 1/\phi(u)$, $H(u,1) = 1$, $H(u,a) = \prod_{l=1}^{a-1} \phi(u+a-2l)$ at $a \ge 2$. It can be expressed through the gamma-function for any a:

$$H(u,a) = 2^{(a-1)N} \prod_{i=1}^{N} \frac{\Gamma\left(\frac{u+a-\xi_i}{2}\right)}{\Gamma\left(\frac{u-a-\xi_i}{2}+1\right)}\,. \tag{11}$$

Given arbitrary functions $T(1,s,u)$, formula (10) gives a solution to the Hirota equation. However, in general this solution does not obey the required analytical and boundary conditions. In particular, the right hand side at $a \ge 2$ has apparent poles at zeros of the function $H(u-s,a)$. Their cancelation is possible if the polynomials $T(1,s,u)$ are chosen in a special way.

We conclude this section by a few words on normalization of the solutions. It is easy to check that the transformation

$$T(a,s,u) \longrightarrow f_0(u+s+a) f_1(u+s-a) f_2(u-s+a) f_3(u-s-a) T(a,s,u), \tag{12}$$

where f_i are arbitrary functions, leaves the form of the Hirota equation unchanged. One may choose certain normalization of the solutions by fixing these functions in one or another way. In our normalization, all the polynomials $T(a,s,u)$ have one and the same degree N equal to the number of sites in the spin chain. This formally includes special cases when one or more zeros of some of these polynomials are placed at infinity, then the degree is actually less then N. Other ways of the normalization are discussed in [3, 9]. In general position, the solutions (6) are irreducible, i.e., $T(a,s,u)$ is not divisible by any polynomial of the form $f_0(u+s+a) f_1(u+s-a) f_2(u-s+a) f_3(u-s-a)$, where at least one of the polynomials $f_i(u)$ has degree greater than 0.

3 The $N = 0$ case: characters of the supergroup $GL(K|M)$

Before proceeding further, it is instructive to consider the case $N = 0$ which appears to be rather meaningful, though simple, and thus provides useful analogies and motivations for dealing with more complicated models. It also emerges as the $u \to \infty$ limit of models with $N > 0$.

At $N=0$ there are no spin degrees of freedom and the T-functions do not depend on u. As is seen from the definition, they coincide with characters of the element $g \in GL(K|M)$ for rectangular representations: $T(a,s,u) = \chi(a,s|g)$. The characters depend on the parameters x_i, y_j entering the matrix g (4). We assume that all x_i, y_j are distinct non-zero numbers. For brevity, we will write $\chi(a,s)$ instead of $\chi(a,s|g)$ when g is fixed.

Let us introduce the rational function

$$w(t) = \frac{\prod_{m=1}^{M}(1-y_m t)}{\prod_{k=1}^{K}(1-x_k t)}\,, \tag{13}$$

where t is an auxiliary variable. As it follows from the character formulas for supergroups [13], $w(t)$ is the generating function of the characters $\chi(1,s)$ while the inverse function $w^{-1}(t)$ is the generating function of the characters $\chi(a,1)$:

$$w(t) = \sum_{s=1}^{\infty} \chi(1,s)\, t^s\,, \quad w^{-1}(t) = \sum_{a=1}^{\infty} (-1)^a \chi(a,1)\, t^a\,. \tag{14}$$

The other characters (super-analogs of Schur functions) are expressed through $\chi(1,s)$ or $\chi(a,1)$ by the Jacobi-Trudi determinant formulas:

$$\chi(a,s) = \det_{1\le i,j\le a} \chi(1,\, s{+}i{-}j) = \det_{1\le i,j\le s} \chi(a{+}i{-}j\,,\, 1)\,. \tag{15}$$

They imply the bilinear relation for the characters of rectangular representations:

$$\chi^2(a,s) = \chi(a+1,s)\chi(a-1,s) + \chi(a,s+1)\chi(a,s-1) \tag{16}$$

which is the u-independent version of eq. (9). It is known as the discrete KdV equation (see, e.g., [18, 19]) written in the bilinear form. Let us also mention the integral representation of the characters

$$\chi(a,s) = \frac{1}{(2\pi i)^a a!} \oint_{|t_1|=1} \cdots \oint_{|t_a|=1} \prod_{1\le j<k\le a} |t_j - t_k|^2 \prod_{n=1}^{a} w(t_n) t_n^{-s-1}\, dt_n \tag{17}$$

where it is assumed that all singularities of the function $w(t)$ are outside the unit circle. This a-fold integral coincides with the partition function of the asymmetric unitary matrix model written through the eigenvalues.

One can formally extend the definition of characters to negative values of a, s by putting them equal to zero. Because $\chi(0,s) = \chi(a,0) = 1$ at $a, s \ge 0$, this is consistent with equation (16) everywhere except at the point $a = s = 0$. To make the equation valid in the whole (a,s) plane, one should put either $\chi(0,n) = 1$ or $\chi(n,0) = 1$ for any $n \in \mathbb{Z}$, all other $\chi(a,s)$ with negative a or s being zero. We choose the former option (consistent with the integral representation (17)). Using representation (15), it is not difficult to show that if $a \ge K+1$ and $s \ge M+1$ then $\chi(a,s) = 0$. Summarizing, we have:

$$\begin{array}{l} \chi(a,s) = 0 \quad \text{if}: \\[1em] \text{(i)}\;\; a<0 \;\; \text{or} \;\; \text{(ii)}\;\; a>0 \;\; \text{and} \;\; s<0, \;\; \text{or} \;\; \text{(iii)}\;\; a>K \;\; \text{and} \;\; s>M. \end{array} \tag{18}$$

We see that the domain where $\chi(a,s)$ do not vanish identically has the shape of a "fat hook" formed by the union of the half-strips $a \ge 0$, $0 \le s \le M$ and $s \ge 0$, $0 \le a \le K$

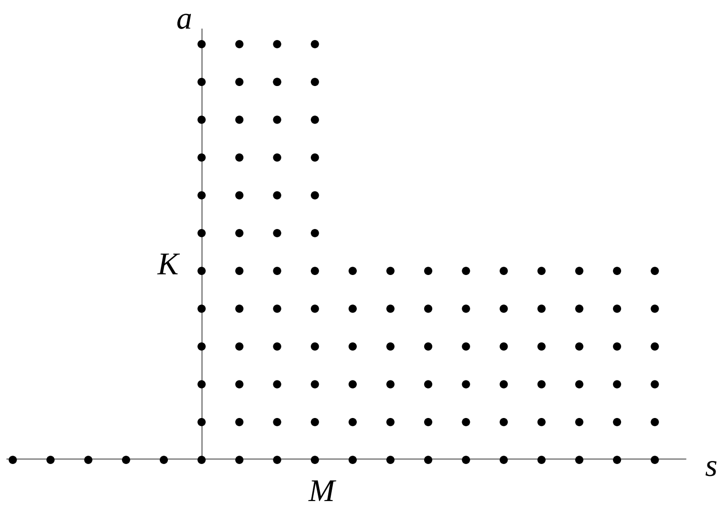

Figure 1: *The domain* $\mathsf{H}(K|M)$ *in the* (a,s)*-lattice at* $K=5$, $M=3$.

together with the horizontal ray $a=0$, $s<0$ from the origin to minus infinity (see Fig. 1). We denote this domain by $\mathsf{H}(K,M)$. We call the boundaries at $a=0$ and $s=0$, $a\geq 0$ *exterior* and the boundaries inside the right upper quadrant *interior* ones.

The characters on the interior boundaries can be found explicitly. A simple calculation shows that $\chi(K,M+n)$ and $\chi(K+n,M)$ at $n\geq 0$ are given by the formulas

$$\chi(K,M+n)=\left(\prod_{k=1}^{K}x_k^n\right)\prod_{i=1}^{K}\prod_{j=1}^{M}(x_i-y_j), \tag{19}$$

$$\chi(K+n,M)=\left(\prod_{m=1}^{M}(-y_m)^n\right)\prod_{i=1}^{K}\prod_{j=1}^{M}(x_i-y_j) \tag{20}$$

(in fact this is a particular case of a more general factorization property, see [20], chapter I, section 3, example 23). Therefore, they are connected by the relation

$$\chi(K,M+n)=(-1)^{nM}(\mathrm{sdet}\,g)^n\,\chi(K+n,M),\quad n\geq 0, \tag{21}$$

where

$$\mathrm{sdet}\,g=\frac{x_1x_2\ldots x_K}{y_1y_2\ldots y_M}\,.$$

Using the determinant identities (Plücker relations) for minors of a rectangular matrix built from the elementary characters $\chi(1,s)$, one can prove the three-term bilinear relations

$$\begin{aligned}&\chi(a+1,s)\tilde\chi(a,s)-\chi(a,s)\tilde\chi(a+1,s)=z\chi(a+1,s-1)\tilde\chi(a,s+1)\,,\\&\chi(a,s+1)\tilde\chi(a,s)-\chi(a,s)\tilde\chi(a,s+1)=z\chi(a+1,s)\tilde\chi(a-1,s+1)\end{aligned} \tag{22}$$

between the characters $\chi(a,s)$ with the generating function $w(t)$ and the characters $\tilde\chi(a,s)$ with the generating function $\tilde w(t)=(1-zt)w(t)$. Both χ and $\tilde\chi$ obey the discrete KdV equation (16) and thus relations (22) generate the Bäcklund transformation for it.

Their meaning is to relate characters of $GL(k|m)$-representations with different k, m. Indeed, let $g_{k,m} \in GL(k|m)$ be the diagonal matrix

$$g_{k,m} = \mathrm{diag}\,(x_1, \ldots, x_k, y_1, \ldots, y_m) \tag{23}$$

obtained from $g = g_{K,M}$ (4) by removing $K-k$ eigenvalues $x_K, x_{K-1}, \ldots, x_{k+1}$ and $M-m$ eigenvalues $y_M, y_{M-1}, \ldots, y_{m+1}$. We put

$$\chi_{k,m}(a,s) = \chi(a,s|g_{k,m}) \tag{24}$$

and, in the same way as in (13), (14), introduce the generating function of the characters $\chi_{k,m}(1,s)$:

$$w_{k,m}(t) = \frac{\prod_{i=1}^{m}(1-y_i t)}{\prod_{j=1}^{k}(1-x_j t)}\,. \tag{25}$$

The obvious recurrence relations

$$\begin{aligned} w_{k-1,m}(t) &= (1-x_k t)\, w_{k,m}(t)\,, \\ w_{k,m+1}(t) &= (1-y_{m+1} t)\, w_{k,m}(t) \end{aligned} \tag{26}$$

allow one to apply equations (22), where one should put $\chi(a,s) = \chi_{k,m}(a,s)$, $\tilde{\chi}(a,s) = \chi_{k-1,m}(a,s)$ at $z = x_k$ or $\chi(a,s) = \chi_{k,m-1}(a,s)$, $\tilde{\chi}(a,s) = \chi_{k,m}(a,s)$ at $z = y_m$.

4 The boundary conditions

In general, the Hirota equation has many solutions of very different natures. The most important additional ingredient which selects the class of solutions relevant to quantum integrable models is the boundary conditions in the variables a, s. Qualitatively, these conditions are the same as those for characters of rectangular representations of supergroups discussed in the previous section, and can be derived by means of a similar reasoning. Furthermore, the explicit formula (1) for the R-matrix implies that the highest coefficient of the polynomial T-function coincides with the corresponding character: $T(a,s,u) = \chi(a,s)u^N + O(u^{N-1})$ as $u \to \infty$.

The distinctive feature of the solutions of our interest is that they are required to vanish identically in some parts of the (a,s)-plane. The domain where they do not do so depends on the symmetry algebra of the quantum model. For the $GL(K|M)$-invariant supersymmetric spin chains it is the domain $\mathsf{H}(K,M)$ introduced in the previous section (Fig. 1). (For the $GL(K)$-invariant spin chains it degenerates to the half-strip $0 \le a \le K$, $s \ge 0$ together with the two rays $s = 0$, $a \ge K$ and $a = 0$, $s \le 0$.) Similarly to (18), we can write:

$$\begin{aligned} &T(a,s,u) = 0 \quad \text{if}: \\ &\text{(i)}\ \ a < 0 \ \text{ or }\ \text{(ii)}\ \ a > 0 \ \text{ and }\ s < 0, \ \text{ or }\ \text{(iii)}\ \ a > K \ \text{ and }\ s > M. \end{aligned} \tag{27}$$

As is easy to see, the shape of $\mathsf{H}(K,M)$ is consistent with the Hirota equation in the whole (a,s)-plane. Although only the points with non-negative a, s have the direct physical

interpretation, it is important to consider the full domain (the "fat hook" complemented by the ray) since otherwise the Hirota equation would brake down at the corner point at the origin.

The boundary values of T-functions have a rather special factorized form fixed by consistency with the Hirota equation. Indeed, on the boundary one of the two terms in the right hand side of eq. (9) vanishes resulting in constraints on the boundary values. For example, at $a = 0$ eq. (9) converts into

$$T(0, s, u+1)T(0, s, u-1) = T(0, s+1, u)T(0, s-1, u)$$

which is a discrete version of the d'Alembert equation with the general solution $T(0, s, u) = f_+(u + s)f_-(u - s)$ where $f_\pm$ are arbitrary functions. In a similar way, $T(a, 0, u) = \tilde{f}_+(u + a)\tilde{f}_-(u - a)$. Equations (7) show that in our normalization $f_+(u) = \tilde{f}_-(u) = 1$ while $f_-(u) = \tilde{f}_+(u) = \phi(u)$, where $\phi(u)$ is defined in (8). A similar factorization holds true on the interior boundaries. However, in general both functions in the product are non-trivial and are to be determined from the TT-relation. The case of the ordinary group $GL(K) = GL(K|0)$, when the vertical interior boundary coincides with the exterior one, is special in this respect. In this case all the boundary values enter as fixed input data. In particular, on the horizontal interior boundary we have $T(K, s, u) = \phi(u + s + K)$.

On the interior boundaries, we impose the condition

$$T(K, M + n, u) = (-1)^{nM}(\mathrm{sdet}\, g)^n\, T(K + n, M, u), \quad n \geq 0, \tag{28}$$

which is consistent with the corresponding condition (21) for characters of the supergroup. For the periodic case (when g is the unity matrix) this condition was pointed out in [7]. Equality (28) means that the values of the T-functions at the points of the interior boundaries equally spaced from the corner point differ by a constant factor only. Note that this condition trivially holds in the case $M = 0$, where these values are fixed from the very beginning as input data. As it was already mentioned, at any $K, M > 0$ they are no longer fixed but are to be determined together with the T-functions inside the domain $\mathsf{H}(K|M)$.

Concluding this section, we note that the parameters x_k, y_m do not enter explicitly neither the Hirota equation nor the boundary conditions for it. This means that the Hirota equation with the fixed boundary conditions on the exterior boundaries of the form (7) has a continuous $(K + M)$-parametric family of polynomial solutions.

5 Auxiliary linear problems

Like almost all known nonlinear integrable equations, the Hirota equation is a compatibility condition for an over-determined system of linear problems [21, 1]. To introduce them, it is convenient to pass to the "chiral" variables

$$\begin{aligned} p &= \tfrac{1}{2}(u - s - a) \\ q &= \tfrac{1}{2}(u + s + a) \\ r &= \tfrac{1}{2}(-u - s + a)\,. \end{aligned} \tag{29}$$

The original variables a, s, u will be referred to as "laboratory" ones. Here are the formulas for the inverse transformation,

$$a = q + r\,, \quad s = -p - r\,, \quad u = p + q \tag{30}$$

and for the transformation of the vector fields:

$$\partial_p = \partial_u - \partial_s\,, \quad \partial_q = \partial_u + \partial_a\,, \quad \partial_r = \partial_a - \partial_s \tag{31}$$

We set $\tau(p, q, r) = T(q+r,\, -p-r,\, p+q)$ and introduce the following linear problems for an auxiliary function $\psi = \psi(p, q, r)$:

$$\begin{aligned} \psi(r+1) + z\frac{\tau(p{+}1, r{+}1)\ \tau}{\tau(p+1)\tau(r+1)}\,\psi &= \psi(p+1) \\ \psi(r+1) - z\frac{\tau(q{+}1, r{+}1)\ \tau}{\tau(q+1)\tau(r+1)}\,\psi &= \psi(q+1) \end{aligned} \tag{32}$$

where z is a parameter. In these formulas, we indicate explicitly only those variables that are subject to shifts. Using these equations, the function $\psi(p+1, q+1)$ can be represented as a linear combination of $\psi(r)$, $\psi(r+1)$ and $\psi(r+2)$ in two different ways. Compatibility of the linear problems means that the results are to be equal. Equating the two expressions we see that the terms proportional to $z^2\psi(r)$ and $\psi(r+2)$ cancel automatically while the terms proportional to $z\psi(r{+}1)$ yield a non-trivial relation (provided $\psi(r+1)$ does not vanish)

$$\begin{aligned} &\frac{\tau(p{+}1, r{+}2)\ \tau(r+1)}{\tau(p{+}1, r{+}1)\tau(r+2)} - \frac{\tau(p{+}1, q{+}1, r{+}1)\tau(p+1)}{\tau(p{+}1, q{+}1)(p{+}1, r{+}1)} \\ = &\frac{\tau(p{+}1, q{+}1, r{+}1)\tau(q+1)}{\tau(p{+}1, q{+}1)(q{+}1, r{+}1)} - \frac{\tau(q{+}1, r{+}2)\ \tau(r+1)}{\tau(q{+}1, r{+}1)\tau(r+2)}\,. \end{aligned}$$

This equality states that the function

$$\frac{\tau(p+1)\tau(q+1, r+1) + \tau(q+1)\tau(p+1, r+1)}{\tau(r+1)\tau(p+1, q+1)}$$

is a periodic function of r with period 1 and an arbitrary function of p, q. Because no special periodicity is implied, we set this function to be r-independent. Therefore, we arrive at the relation

$$\tau(p+1)\tau(q+1, r+1) + \tau(q+1)\tau(p+1, r+1) = h(2p, 2q)\tau(r+1)\tau(p+1, q+1)\,,$$

where h can be an arbitrary function of p and q. In the original variables this equation reads

$$T(a+1)T(a-1) + T(s+1)T(s-1) = h(u{-}s{-}a, u{+}s{+}a)\, T(u+1)T(u-1)\,.$$

From the boundary conditions (7) at $a = 0$ or $s = 0$ it follows that $h = 1$ and we obtain the Hirota equation (9). Note that the parameter z entering the linear problems disappears from the non-linear equation. In fact this is clear from the very beginning

because z can be eliminated from equations (32) by the transformation $\psi \to z^{p+q+r}\psi$. Nevertheless, we keep this parameter because it will be important in what follows.

An advantage of the "chiral" variables is their separation in the linear problems: the first problem does not involve q while the second one does not involve p. However, in contrast to the "laboratory" variables a, s, u, they have no immediate physical meaning. Coming back to the "laboratory" variables, we set $\psi(p, q, r) = \Psi(q + r,\ -p - r,\ p + q)$ and rewrite the linear problems (32) in the form

$$
\begin{aligned}
&\Psi(a,s,u) + z\frac{T(a-1,s+1,u)T(a,s-1,u+1)}{T(a,s,u)\,T(a-1,s,u+1)}\,\Psi(a-1,s+1,u) = \Psi(a-1,s,u+1) \\
&\Psi(a,s,u) - z\frac{T(a-1,s+1,u)T(a+1,s,u+1)}{T(a,s,u)\,T(a,s+1,u+1)}\,\Psi(a-1,s+1,u) = \Psi(a,s+1,u+1)\,.
\end{aligned}
\tag{33}
$$

In general, compatibility of linear problems implies existence of a continuous family of common solutions. As we have seen, the structure of the coefficient functions in our case is such that the compatibility is equivalent to the existence of *at least one* common solution (see [22], where this fact was pointed out in another context).

Because the T-functions can vanish identically at some a, s, we eliminate the denominators by passing to the new auxiliary function $F = T\Psi$, in terms of which we have

$$
\begin{aligned}
&T(a-1,s,u+1)F(a,s,u) + zT(a,s-1,u+1)F(a-1,s+1,u) = T(a,s,u)F(a-1,s,u+1) \\
&T(a,s+1,u+1)F(a,s,u) - zT(a+1,s,u+1)F(a-1,s+1,u) = T(a,s,u)F(a,s+1,u+1)\,.
\end{aligned}
\tag{34}
$$

Note that the second equation can be obtained from the first one by the transformation $T(a,s,u) \longrightarrow (-1)^{as}T(-s,-a,u)$ (and the same for F) which leaves the Hirota equation invariant. However, the Hirota equation written for the function $\tilde{T}(a,s,u) = (-1)^{\frac{1}{2}(a^2+s^2)}T(a,s,u)$ is form-invariant with respect to a larger symmetry group consisting of any permutations and changing signs of the variables a, s, u but the system of the linear problems (34) is not. In fact the symmetry is realized in an implicit way. To make it explicit, we write the pair of equations (34) in a matrix form,

$$
\begin{aligned}
&\begin{pmatrix} T(a-1,s,u) & zT(a,s-1,u) \\ T(a,s+1,u) & -zT(a+1,s,u) \end{pmatrix} \begin{pmatrix} F(a,\ s,\ u-1) \\ F(a-1,s+1,u-1) \end{pmatrix} \\
&\qquad = \ T(a,s,u-1) \begin{pmatrix} F(a-1,s,u) \\ F(a,s+1,u) \end{pmatrix},
\end{aligned}
\tag{35}
$$

and multiply both sides by the matrix inverse to the one in the left hand side. Using the TT-relation, we get another pair of linear problems,

$$
\begin{aligned}
&T(a+1,s+1,u)F(a,s,u) - zT(a+1,s,u+1)F(a,s+1,u-1) = T(a,s,u)F(a+1,s+1,u) \\
&T(a,s,u+1)F(a,s,u-1) - T(a,\ s-1,u)\,F(a,s+1,u) = T(a+1,\ s,u)F(a-1,s,u)
\end{aligned}
\tag{36}
$$

which are equivalent to (and thus compatible with) the pair (34) by construction. The set of four linear problems (34), (36) possesses the required symmetry. The Hirota equation can be derived as a compatibility condition for any two linear problems of these four, and the other two hold automatically. The four linear equations can be combined into a single matrix equation

$$\mathbb{T}(a,s,u)\begin{pmatrix} F(a-1,\, s,\, u) \\ F(a,\, s+1,\, u) \\ F(a,\, s,\, u-1) \\ zF(a-1, s+1, u-1) \end{pmatrix} = 0, \tag{37}$$

where $\mathbb{T}(a,s,u)$ is the antisymmetric matrix

$$\mathbb{T}(a,s,u) = \begin{pmatrix} 0 & T(a,s,u-1) & -T(a,s+1,u) & T(a+1,s,u) \\ -T(a,s,u-1) & 0 & T(a-1,s,u) & T(a,s-1,u) \\ T(a,s+1,u) & -T(a-1,s,u) & 0 & -T(a,s,u+1) \\ -T(a+1,s,u) & -T(a,s-1,u) & T(a,s,u+1) & 0 \end{pmatrix}. \tag{38}$$

The Hirota equation implies that its determinant vanishes and rank of this matrix equals 2. The symmetric form of the linear problems for the Hirota equation was suggested in [23]. For more details on the linear problems and their symmetries see [21, 19, 23, 24].

6 Bäcklund transformations

There is a remarkable duality between $T(a,s,u)$ and $F(a,s,u)$ [21, 1]: one can exchange their roles and treat eqs. (34) as an over-determined system of linear problems for the function T with coefficients F. Their compatibility condition is the same Hirota equation for F:

$$F(a,s,u+1)F(a,s,u-1) = F(a,s+1,u)F(a,s-1,u)+F(a+1,s,u)F(a-1,s,u)\,. \tag{39}$$

We thus conclude that any solution to the linear problems (34), where the T-function obeys the Hirota equation, provides an (auto) Bäcklund transformation, i.e., a transformation that sends a solution of the nonlinear integrable equation to another solution of the same equation.

Let us rewrite the linear problems (34) changing the order of the terms and shifting the variables:

$$\begin{aligned} &T(a+1,s,u)F(a,s,u+1) - T(a,s,u+1)F(a+1,s,u) = zT(a+1,s-1,u+1)F(a,s+1,u) \\ &T(a,s+1,u+1)F(a,s,u) - T(a,s,u)F(a,s+1,u+1) = zT(a+1,s,u+1)F(a-1,s+1,u). \end{aligned} \tag{40}$$

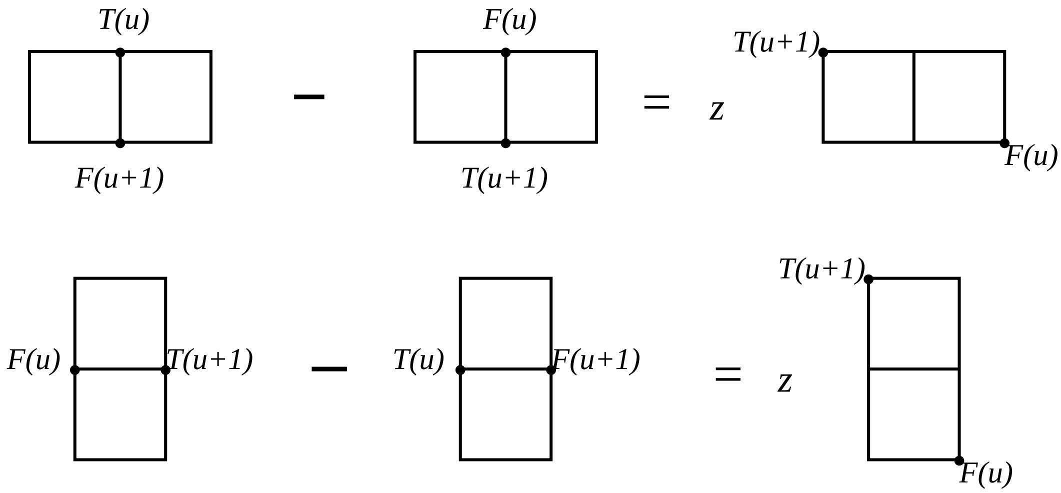

Figure 2: *The graphical representation of equations (40) in the (a,s)-lattice. Here a and s coordinates correspond to the vertical and horizontal axis respectively.*

These equations are graphically represented in Fig. 2 in the (a,s)-plane. They constitute the Bäcklund transformation $T \to F$ in the bilinear form [8, 21]. Given a family of polynomials $T(a,s,u)$ obeying the Hirota equation, one may pose the problem of finding polynomial solutions to equations (40). It is easy to see that these equations are not compatible with the boundary conditions for $F(a,s,u)$ and $T(a,s,u)$ of the "fat hook" type with the same K and M. Indeed, applying these equations in the corner point of the interior boundary, one sees that if K, M for T and F are the same, then the boundary values must vanish identically. However, it is straightforward to verify that equations (40) are compatible with the boundary conditions of the following two types:

$$\begin{aligned} &F(a,s,u) = 0 \quad \text{if :} \\ &\text{(i)} \;\; a<0 \quad \text{or} \quad \text{(ii)} \;\; a>0 \;\; \text{and} \;\; s<0, \quad \text{or} \quad \text{(iii)} \;\; a>K-1 \;\; \text{and} \;\; s>M, \end{aligned} \tag{41}$$

or

$$\begin{aligned} &F(a,s,u) = 0 \quad \text{if :} \\ &\text{(i)} \;\; a<0 \quad \text{or} \quad \text{(ii)} \;\; a>0 \;\; \text{and} \;\; s<0, \quad \text{or} \quad \text{(iii)} \;\; a>K \;\; \text{and} \;\; s>M+1\,. \end{aligned} \tag{42}$$

They are again of the "fat hook" type but with the shifts $K \to K-1$ or $M \to M+1$. We refer to the corresponding transformations as BT_1^- and BT_2^+: $F(a,s,u) = \mathrm{BT}_1^-(T(a,s,u))$ for (41) and $F(a,s,u) = \mathrm{BT}_2^+(T(a,s,u))$ for (42). They depend on the parameter z. The same formulas (40) define inverse transformations $\mathrm{BT}_1^+ = (\mathrm{BT}_1^-)^{-1}$ and $\mathrm{BT}_2^- = (\mathrm{BT}_2^+)^{-1}$ if one treats them as linear equations for T with given F. In a more explicit way, the transformations $\mathrm{BT}_1^\pm$, $\mathrm{BT}_2^\pm$ are defined by formulas (47), (48) below.

Repeating these transformations several times, we arrive at the hierarchy of functions $T_{k,m}(a,s,u)$ ($k = 0,1,\ldots,K$, $m = 0,1,\ldots,M$) such that:

a) They obey the Hirota equation in a,s,u for any k,m;

b) They are connected by the Bäcklund transformations

$$
\begin{aligned}
&T_{k-1,m}(a,s,u) = \mathrm{BT}_1^-(T_{k,m}(a,s,u)),\\
&T_{k,m-1}(a,s,u) = \mathrm{BT}_2^-(T_{k,m}(a,s,u));
\end{aligned}
\tag{43}
$$

c) At $k = K$, $m = M$ we have

$$
T_{K,M}(a,s,u) = T(a,s,u)\,.
$$

For the irreducible polynomial solutions this list should be supplemented by the "initial condition" $T_{0,0}(a,s,u) = 1$ at $a = 0$ or at $s = 0$, $a > 0$ and $T_{0,0}(a,s,u) = 0$ otherwise. The levels of the hierarchy are labeled by the pair of numbers k, m. The lowest level is $0, 0$ while the highest one is K, M. We shall say that the T-functions $T_{k,m}(a,s,u)$ belong to the level k, m.

It is important to note that the parameter z involved in the definition of the Bäcklund transformations can be different for transformations of the two types introduced above as well as for successive transformations of the same type. We choose these parameters to be eigenvalues of the matrix g: $z = x_k$ for the transition $(k,m) \to (k-1,m)$ and $z = y_m$ for $(k,m) \to (k,m-1)$. More precisely, the transformation BT_1^- is written in the form

$$
\begin{aligned}
&T_{k,m}(a+1,s,u)T_{k-1,m}(a,s,u+1) - T_{k,m}(a,s,u+1)T_{k-1,m}(a+1,s,u)\\
&\qquad = \; x_k\, T_{k,m}(a+1,s-1,u+1)T_{k-1,m}(a,s+1,u),\\
&T_{k,m}(a,s+1,u+1)T_{k-1,m}(a,s,u) - T_{k,m}(a,s,u)T_{k-1,m}(a,s+1,u+1)\\
&\qquad = \; x_k\, T_{k,m}(a+1,s,u+1)T_{k-1,m}(a-1,s+1,u),
\end{aligned}
\tag{44}
$$

while BT_2^- in the form

$$
\begin{aligned}
&T_{k,m-1}(a+1,s,u)T_{k,m}(a,s,u+1) - T_{k,m-1}(a,s,u+1)T_{k,m}(a+1,s,u)\\
&\qquad = \; y_m\, T_{k,m-1}(a+1,s-1,u+1)T_{k,m}(a,s+1,u),\\
&T_{k,m-1}(a,s+1,u+1)T_{k,m}(a,s,u) - T_{k,m-1}(a,s,u)T_{k,m}(a,s+1,u+1)\\
&\qquad = \; y_m\, T_{k,m-1}(a+1,s,u+1)T_{k,m}(a-1,s+1,u)
\end{aligned}
\tag{45}
$$

($k = 1, \ldots, K$, $m = 1, \ldots, M$). If one ignores the u-dependence (which disappears in the $u \to \infty$ limit), then these formulas become the bilinear relations between characters mentioned in Section 3 (see equations (22) and comments after them). It is easy to notice that each of the equations in (44), (45) is actually a dynamical equation for a function of three variables rather than five. For example, the first equation in (44) acts in the subspaces $m = \mathrm{const}$ and $u + s + a = \mathrm{const}$. Upon restriction to the corresponding three-dimensional hyperplanes in the linear space with coordinates a, s, u, k, m, each of these equations can be put in the standard Hirota form by a linear change of variables.

For completeness, let us give a symmetric description of the Bäcklund transformations through the matrix equations of the form (37). For the direct and inverse transformations we need to introduce two antisymmetric 4×4 matrices $\mathbb{T}^{(\pm 1)}(a,s,u)$ of the type (38):

$$\mathbb{T}^{(\varepsilon)}(a,s,u) = \begin{pmatrix} 0 & T(a,s,u-\varepsilon) & -T(a,s+\varepsilon,u) & T(a+\varepsilon,s,u) \\ -T(a,s,u-\varepsilon) & 0 & T(a-\varepsilon,s,u) & T(a,s-\varepsilon,u) \\ T(a,s+\varepsilon,u) & -T(a-\varepsilon,s,u) & 0 & -T(a,s,u+\varepsilon) \\ -T(a+\varepsilon,s,u) & -T(a,s-\varepsilon,u) & T(a,s,u+\varepsilon) & 0 \end{pmatrix} \tag{46}$$

where $\varepsilon=\pm 1$. Then the transformations $T_{k-1,m}=\mathrm{BT}_1^{-}(T_{k,m})$, $T_{k,m+1}=\mathrm{BT}_2^{+}(T_{k,m})$ are defined by the matrix equation

$$\mathbb{T}^{(+1)}_{k,m}(a,s,u)\begin{pmatrix} T_{k-1,m}(a-1,\ s,\ u) & T_{k,m+1}(a-1,\ s,\ u) \\ T_{k-1,m}(a,\ s+1,\ u) & T_{k,m+1}(a,\ s+1,\ u) \\ T_{k-1,m}(a,\ s,\ u-1) & T_{k,m+1}(a,\ s,\ u-1) \\ x_k T_{k-1,m}(a–1,s+1,u–1) & y_{m+1}T_{k,m+1}(a–1,s+1,u–1) \end{pmatrix} = 0, \tag{47}$$

and the inverse transformations $T_{k+1,m}=\mathrm{BT}_1^{+}(T_{k,m})$, $T_{k,m-1}=\mathrm{BT}_2^{-}(T_{k,m})$ are defined by the equation

$$\mathbb{T}^{(-1)}_{k,m}(a,s,u)\begin{pmatrix} T_{k+1,m}(a+1,\ s,\ u) & T_{k,m-1}(a+1,\ s,\ u) \\ T_{k+1,m}(a,\ s-1,\ u) & T_{k,m-1}(a,\ s-1,\ u) \\ T_{k+1,m}(a,\ s,\ u+1) & T_{k,m-1}(a,\ s,\ u+1) \\ x_{k+1} T_{k+1,m}(a+1,s–1,u+1) & y_{m}T_{k,m-1}(a+1,s–1,u+1) \end{pmatrix} = 0, \tag{48}$$

where $\mathbb{T}^{(\pm)}_{k,m}(a,s,u)$ are the matrices (46) with entries at the level k,m.

Moreover, a careful analysis of the equations (44)-(48) shows that boundary values of the T-functions at each level k,m are subject to the same relations as at the highest level. On the exterior boundaries, the T-functions have the specific form similar to (7), i.e., $T_{k,m}(0,s,u)$ is a function of $u-s$ while $T_{k,m}(a,0,u)$ is the same function of $u+a$. Let us introduce the special notation for them:

$$T_{k,m}(0,s,u)=Q_{k,m}(u-s)\,,\quad T_{k,m}(a,0,u)=Q_{k,m}(u+a)\,. \tag{49}$$

The polynomials $Q_{k,m}(u)$ play a very important role. They will be identified with eigenvalues of Baxter's Q-operators. The polynomial $Q_{K,M}(u)=\phi(u)$ is a fixed input data which determines the model. The polynomials $Q_{k,m}(u)$ at lower levels are to be found in the course of solution. In analogy with (8), we fix their highest coefficients to be 1. Applying (44), (45) on the interior boundaries, we conclude that if (28) is valid at the

highest level, then T-functions on the interior boundaries at each level k, m are connected by a similar relation:

$$T_{k,m}(k, m+n, u) = (-1)^{nm} (\mathrm{sdet}\, g_{k,m})^n \, T_{k,m}(k+n, m, u), \quad n \geq 0. \tag{50}$$

The matrix $g_{k,m} \in GL(k|m)$ is defined in (23) and $\mathrm{sdet}\, g_{k,m} = x_1 \dots x_k/(y_1 \dots y_m)$.

We see now that by applying BT_1^- and BT_2^- to a solution of the Hirota equation in the domain $\mathsf{H}(K|M)$ with the boundary conditions (27), we can successively transform it to the trivial solution in the degenerate domain $\mathsf{H}(0|0)$. This "undressing procedure" allows one to construct solutions to the original problem, as it will be shown below.

7 Recurrence relations for the operator generating series

The Bäcklund transformations from the previous section can be reformulated in the operator form as recurrence relations for difference operators of infinite order. Let us consider the following operator:

$$\mathcal{W}(u) = \sum_{s \geq 0} \frac{T(1, s, u+s-1)}{\phi(u)} \, e^{2s\partial_u} \tag{51}$$

(the common denominator is introduced for normalization). It serves as a non-commutative generating series for the T-functions $T(1, s, u)$. Similar objects can be introduced at any level k, m:

$$\mathcal{W}_{k,m}(u) = \sum_{s \geq 0} \frac{T_{k,m}(1, s, u+s-1)}{Q_{k,m}(u)} \, e^{2s\partial_u} \, . \tag{52}$$

It is the generating series for the functions $T_{k,m}(1, s, u)$. Clearly, $\mathcal{W}_{0,0}(u) = 1$ (recall that $T_{0,0}(1, s, u) = 0$ unless $s = 0$ and $T_{0,0}(1, 0, u) = Q_{0,0}(u+1) = 1$). We also note that the series formally inverse to (52) generates the T-functions $T_{k,m}(a, 1, u)$:

$$\mathcal{W}_{k,m}^{-1}(u) = \sum_{a \geq 0} (-1)^a \, e^{2a\partial_u} \, \frac{T_{k,m}(a, 1, u-a-1)}{Q_{k,m}(u-2)} \, . \tag{53}$$

For the proof, see [9].

Set

$$X_{k,m}(u) = x_k \, \frac{Q_{k,m}(u+2) \, Q_{k-1,m}(u-2)}{Q_{k,m}(u) \; Q_{k-1,m}(u)} \, , \tag{54}$$

$$Y_{k,m}(u) = y_m \, \frac{Q_{k,m-1}(u+2) \, Q_{k,m}(u-2)}{Q_{k,m-1}(u) \; Q_{k,m}(u)} \, . \tag{55}$$

Using the linear problems at $a = 0$, it is a straightforward calculation to prove the following recurrence relations for the operators $\mathcal{W}_{k,m}(u)$:

$$\begin{aligned} \mathcal{W}_{k-1,\,m}(u) &= \left(1 - X_{k,m}(u) e^{2\partial_u}\right) \mathcal{W}_{k,m}(u) \, , \\ \mathcal{W}_{k,\,m+1}(u) &= \left(1 - Y_{k,m+1}(u) e^{2\partial_u}\right) \mathcal{W}_{k,m}(u) \, . \end{aligned} \tag{56}$$

These formulas are operator (and u-dependent) analogs of (26). The shift operator $\mathbf{t} = e^{2\partial_u}$ plays the role of the variable t while the variable u is absent in (26). Notice also that $X_{k,m}(u)$, $Y_{k,m}(u)$ turn into x_k, y_m in the limit $u \to \infty$.

Here we present some details of the proof (see also [9], where a slightly different version of the recurrence relations is proved). Consider the first relation. We have:

$$\mathcal{W}_{k-1,m}(u) - \mathcal{W}_{k,m}(u) = \sum_{s\geq 0}\left[\frac{T_{k-1,m}(1,s,u+s-1)}{Q_{k-1,m}(u)} - \frac{T_{k,m}(1,s,u+s-1)}{Q_{k,m}(u)}\right] e^{2s\partial_u} . \quad (57)$$

To transform the expression in the square brackets, we rewrite the first equation in (44) at $a = 0$ in the form

$$\begin{aligned}&\frac{T_{k,m}(1,s,u+s-1)}{Q_{k,m}(u)} - \frac{T_{k-1,m}(1,s,u+s-1)}{Q_{k-1,m}(u)} \\ =\quad & x_k\,\frac{Q_{k,m}(u+2)Q_{k-1,m}(u-2)}{Q_{k,m}(u)Q_{k-1,m}(u)}\,\frac{T_{k,m}(1,s-1,u+s)}{Q_{k,m}(u+2)}\end{aligned}$$

and continue the equality:

$$\mathcal{W}_{k-1,m}(u) - \mathcal{W}_{k,m}(u) = -\,X_{k,m}(u)\sum_{s\geq 0}\frac{T_{k,m}(1,s-1,u+s)}{Q_{k,m}(u+2)}\;e^{2s\partial_u} . \quad (58)$$

Because $T_{k,m}(1,-1,u) = 0$, the sum in right hand side can be written in the form

$$\sum_{s\geq 0}\frac{T_{k,m}(1,s-1,u+s)}{Q_{k,m}(u+2)}\;e^{2s\partial_u} = e^{2\partial_u}\sum_{s\geq 0}\frac{T_{k,m}(1,s,u+s-1)}{Q_{k,m}(u)} = e^{2\partial_u}\mathcal{W}_{k,m}(u)\,,$$

and the first recurrence relation is proved. The proof of the second one is completely similar.

8 Factorization formulas and TQ-relations

The recurrence relations established in the previous section allow one to represent the operator generating series (51) in a closed factorized form, where each factor contains the Q-functions only. Namely, $\mathcal{W}_{K,M}(u)$ can be obtained as a result of successive application of the recurrence relations (56) to $\mathcal{W}_{0,0}(u) = 1$. In this way, moving first in the m-direction from $(0,0)$ to $(0,M)$ and then in the k-direction from $(0,M)$ to (K,M), we get:

$$\mathcal{W}_{K,M}(u) = \prod_{K\geq k\geq 1}^{\longleftarrow}\Big(1 - X_{k,M}(u)e^{2\partial_u}\Big)^{-1}\cdot\prod_{M\geq m\geq 1}^{\longleftarrow}\Big(1 - Y_{0,m}(u)e^{2\partial_u}\Big) \quad (59)$$

where the ordered product is defined as

$$\prod_{J\geq i\geq I}^{\longleftarrow} A_i = A_J\,A_{J-1}\,\ldots\,A_{I+1}\,A_I\,.$$

Applying the recurrence relations in the different order (k-direction first, m-direction next), we arrive at a different but equivalent representation:

$$\mathcal{W}_{K,M}(u) = \prod_{M \geq m \geq 1}^{\leftarrow} \Big(1 - Y_{K,m}(u) e^{2\partial_u}\Big) \cdot \prod_{K \geq k \geq 1}^{\leftarrow} \Big(1 - X_{k,0}(u) e^{2\partial_u}\Big)^{-1} . \tag{60}$$

In fact one can apply the relations (56) in any other order determined by a chosen zigzag path from the point $(0,0)$ to the point (K,M). Each step in the k-direction, $(k,m) \to (k+1,m)$, brings the factor $\Big(1 - X_{k+1,m}(u) e^{2\partial_u}\Big)^{-1}$ while each step in the m-direction, $(k,m) \to (k,m+1)$, brings the factor $\Big(1 - Y_{k,m+1}(u) e^{2\partial_u}\Big)$ which are to be multiplied according to the order of the steps. This yields many other ways to factorize the operator $\mathcal{W}_{K,M}(u)$. Their equivalence follows from the compatibility of the recurrence relations (56) which is expressed by the discrete "zero curvature" condition

$$\Big(1 - Y_{k-1,m+1}(u) e^{2\partial_u}\Big)\Big(1 - X_{k,m}(u) e^{2\partial_u}\Big) = \Big(1 - X_{k,m+1}(u) e^{2\partial_u}\Big)\Big(1 - Y_{k,m+1}(u) e^{2\partial_u}\Big) \tag{61}$$

on the (k,m)-lattice. The two sides of this equality correspond to two different ways to obtain $\mathcal{W}_{k-1,m+1}(u)$ from $\mathcal{W}_{k,m}(u)$.

The equalities (59) and (60) as well as the similar equalities with different orderings are generalized Baxter's TQ-relations in a generating form. Equating coefficients in front of different powers of the operator $e^{2\partial_u}$, one obtains expressions for the T-functions $T(1,s,u) = T_{K,M}(1,s,u)$ through $X_{k,m}$, $Y_{k,m}$ and thus through the Q-functions $Q_{k,m}(u)$ with $1 \leq k \leq K$, $1 \leq m \leq M$. For example, the simplest TQ-relation contained in (59) has the form

$$\frac{T_{K,M}(1,1,u)}{Q_{K,M}(u)} = \sum_{k=1}^{K} X_{k,M}(u) - \sum_{m=1}^{M} Y_{0,m}(u) \,, \tag{62}$$

where $X_{k,M}(u)$, $Y_{0,m}(u)$ are to be expressed through the Q-functions according to (54), (55). The zeros of the latter are to be constrained by the system of Bethe equations derived in the next section.

9 QQ-relation and Bethe equations

Our starting point in this section is the discrete zero curvature condition (61). Comparing coefficients in front of different powers of the shift operator, we note that

$$Y_{k-1,m+1}(u) X_{k,m}(u+2) = X_{k,m+1}(u) Y_{k,m+1}(u+2)$$

holds identically, and thus get the only non-trivial relation

$$Y_{k-1,m+1}(u) + X_{k,m}(u) = X_{k,m+1}(u) + Y_{k,m+1}(u) \,,$$

which after the substitution (54), (55) becomes a functional equation for the Q-functions. As a simple calculation shows, it is equivalent to the following bilinear equation (the "QQ-relation" [9]):

$$\begin{aligned} & x_k Q_{k-1,m-1}(u) Q_{k,m}(u+2) - y_m Q_{k,m}(u) Q_{k-1,m-1}(u+2) \\ = \quad & (x_k - y_m)\, Q_{k-1,m}(u) Q_{k,m-1}(u+2) \end{aligned} \tag{63}$$

for the polynomial functions

$$Q_{k,m}(u) = \prod_{j=1}^{N_{k,m}} (u - u_j^{(k,m)})\,. \tag{64}$$

Let us remark that the QQ-relation (63) can be recast to the standard form of the Hirota bilinear difference equation in "chiral" variables $u, m, -k$ by passing to the function

$$\mathcal{Q}_{k,m}(u) = a_{k,m} e^{(\beta_k+\gamma_m)u}\, Q_{k,m}(u)\,, \tag{65}$$

where the new parameters β_k and γ_m are related to the x_k, y_m by the formulas

$$x_k = e^{2(\beta_k-\beta_{k-1})}\,, \quad y_m = e^{2(\gamma_{m-1}-\gamma_m)}\,. \tag{66}$$

They are fixed uniquely by putting $\beta_0 = \gamma_0 = 0$. This transformation eliminates the coefficients x_k, y_m in (63) (as well as in (54), (55)) and, with a proper choice of $a_{k,m}$, the QQ-relation acquires the coefficient-free form

$$\mathcal{Q}_{k-1,m-1}(u)\mathcal{Q}_{k,m}(u+2) - \mathcal{Q}_{k,m}(u)\mathcal{Q}_{k-1,m-1}(u+2) = \mathcal{Q}_{k-1,m}(u)\mathcal{Q}_{k,m-1}(u+2) \tag{67}$$

suggested in [9].

The QQ-relation (63) provides the easiest and the most transparent way to derive Bethe equations for roots of the polynomials $Q_{k,m}(u)$. Putting u in (63) successively equal to the roots of each Q-function entering the equation, one obtains a number of relations which, after some rearranging, can be written in the form

$$\frac{Q_{k-1,m}\left(u_j^{(k,m)}\right) Q_{k,m}\left(u_j^{(k,m)}-2\right) Q_{k+1,m}\left(u_j^{(k,m)}+2\right)}{Q_{k-1,m}\left(u_j^{(k,m)}-2\right) Q_{k,m}\left(u_j^{(k,m)}+2\right) Q_{k+1,m}\left(u_j^{(k,m)}\right)} = -\,\frac{x_k}{x_{k+1}}\,, \tag{68}$$

$$\frac{Q_{k,m+1}\left(u_j^{(k,m)}\right) Q_{k,m}\left(u_j^{(k,m)}-2\right) Q_{k,m-1}\left(u_j^{(k,m)}+2\right)}{Q_{k,m+1}\left(u_j^{(k,m)}-2\right) Q_{k,m}\left(u_j^{(k,m)}+2\right) Q_{k,m-1}\left(u_j^{(k,m)}\right)} = -\,\frac{y_{m+1}}{y_m}\,, \tag{69}$$

$$\frac{Q_{k+1,m}\left(u_j^{(k,m)}\right) Q_{k,m-1}\left(u_j^{(k,m)}+2\right)}{Q_{k+1,m}\left(u_j^{(k,m)}+2\right) Q_{k,m-1}\left(u_j^{(k,m)}\right)} = \frac{x_{k+1}}{y_m}\,, \tag{70}$$

$$\frac{Q_{k,m+1}\left(u_j^{(k,m)}\right) Q_{k-1,m}\left(u_j^{(k,m)}-2\right)}{Q_{k,m+1}\left(u_j^{(k,m)}-2\right) Q_{k-1,m}\left(u_j^{(k,m)}\right)} = \frac{y_{m+1}}{x_k}\,. \tag{71}$$

They hold inside the $K \times M$ rectangle in the (k,m)-lattice and serve as elementary building blocks for systems of Bethe equations. Each such system corresponds to a zigzag "undressing" path from (K,M) to $(0,0)$. On the parts of the path $(k+1,m) \to (k,m) \to (k-1,m)$, $(k,m+1) \to (k,m) \to (k,m-1)$, $(k+1,m) \to (k,m) \to (k,m-1)$ and $(k,m+1) \to (k,m) \to (k-1,m)$ one uses (68), (69), (70) and (71) respectively. These systems are different but equivalent. For a more detailed discussion on this point,

see [9]. As an example, we give here the chain of the Bethe equations for the simplest path $(K, M) \longrightarrow (0, M) \longrightarrow (0, 0)$. Moving from (K, M) to $(0, M)$, we have the equations

$$\frac{Q_{k-1,M}\left(u_j^{(k,M)}\right) Q_{k,M}\left(u_j^{(k,M)}-2\right) Q_{k+1,M}\left(u_j^{(k,M)}+2\right)}{Q_{k-1,M}\left(u_j^{(k,M)}-2\right) Q_{k,M}\left(u_j^{(k,M)}+2\right) Q_{k+1,M}\left(u_j^{(k,M)}\right)} = -\frac{x_k}{x_{k+1}}\,, \tag{72}$$

where $k = 1, \ldots, K-1$. They agree with the chain of Bethe equations presented in [25] for the bosonic case. At the turning point, the equation is

$$\frac{Q_{1,M}\left(u_j^{(0,M)}\right) Q_{0,M-1}\left(u_j^{(0,M)}+2\right)}{Q_{1,M}\left(u_j^{(0,M)}+2\right) Q_{0,M-1}\left(u_j^{(0,M)}\right)} = \frac{x_1}{y_M}\,. \tag{73}$$

Finally, moving from $(0, M)$ to $(0, 0)$, we have the equations

$$\frac{Q_{0,m+1}\left(u_j^{(0,m)}\right) Q_{0,m}\left(u_j^{(0,m)}-2\right) Q_{0,m-1}\left(u_j^{(0,m)}+2\right)}{Q_{0,m+1}\left(u_j^{(0,m)}-2\right) Q_{0,m}\left(u_j^{(0,m)}+2\right) Q_{0,m-1}\left(u_j^{(0,m)}\right)} = -\frac{y_{m+1}}{y_m}\,, \tag{74}$$

where $m = 1, 2, \ldots, M-1$.

10 Conclusion

We have obtained a solution of the TT-relation (the Hirota equation) obeying all the required boundary and analytic conditions. The solution is given by the determinant formula (10), where the polynomials $T(1, s, u)$ are determined by the expansion (51) of the factorized operator (59). The coefficients of the latter are expressed through the Q-functions via equations (54), (55), where the roots of the polynomials $Q_{k,m}(u)$ are constrained by Bethe equations which result from the bilinear QQ-relation (63). It should be noted that the solution is not unique. Given boundary conditions, there is a finite set of solutions corresponding to different quantum states of the generalized spin chain.

We emphasize that this solution, typical for quantum problems solvable by Bethe ansatz, has been obtained by purely classical methods of the theory of soliton equations on the lattice. The key role is played by Bäcklund transformations for the Hirota difference equation. From the viewpoint of the theory of classical soliton equations, our method consists in constructing a chain of successive Bäcklund transformations which reduces the problem to a trivial one. Each transformation of this chain involves a continuous parameter (the classical spectral parameter) which is identified with an eigenvalue of the matrix which defines the twisted boundary conditions in the quantum integrable model.

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